Viability for differential equations driven by fractional Brownian motion

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Abstract

In this paper we prove a viability result for multidimensional, time dependent, stochastic differential equations driven by fractional Brownian motion with Hurst parameter$\frac{1}{2} < H < 1$, using pathwise approach. The sufficient condition is also an alternative global existence result for the fractional differential equations with restrictions on the state.

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1 Introduction

Let $B = \{B_t, t \geq 0\}$ be a fractional Brownian motion (fBm) of Hurst parameter $H \in (0, 1)$. That is, B is a centered Gaussian process with the covariance function (see [12])

$$R_H(s,t) = \mathbb{E}\left(B_s B_t\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right). \tag{1}$$

Notice that if $H = \frac{1}{2}$, the process B is a standard Brownian motion, but if $H \neq \frac{1}{2}$, it does not have independent increments. It clearly follows that $\mathbb{E}|B_t - B_s|^2 = |t-s|^{2H}$. As a consequence, the process B has $\alpha-$Holder continuous paths for all $\alpha \in (0, H)$.

The definition of stochastic integrals with respect to the fractional Brownian motion has been investigated by several authors.

Essentially two different types of integrals can be defined:

- The divergence integral (or Skorohod integral) with respect to fBm is defined as the adjoint of the derivative operator in the framework of the Malliavin calculus. This approach was introduced by Decreusefond and Üstünel [7] and developed by Carmona and Coutin [6], Duncan, Hu and Pasik-Duncan [8], Alos, Mazet and Nualart [1], Hu and Øksendal [10], among others. This stochastic integral can be expressed as the limit of Riemann sums defined using Wick products and satisfies the zero mean property

- The pathwise Riemann-Stieltjes integral $\int_0^T u_s dB_s^H$ which exists if the stochastic process $(u_t)_{t\in[0,T]}$ has continuous paths of order $\alpha > 1-H$, is a consequence of the result of Young [17]. Zähle has defined in [18] and [19] this integral for processes with paths in a fractional Sobolev type space. In this paper we will follow this last approach.

The aim of this paper is to state necessary and sufficient conditions that guarantee that the solution of a given (forward) stochastic differential equation driven by the fractional Brownian motion B with Hurst parameter $1/2 < H < 1$ (in short: f-SDE), $\mathbb{P}-a.s.\ \omega \in \Omega$

$$X_s^{t,x} = x + \int_t^{s\vee t} b(r, X_r^{t,x})dr + \int_t^{s\vee t} \sigma(r, X_r^{t,x})dB_r^H, \quad s \in [t,T], \tag{2}$$

$(t,x) \in [0,T] \times \mathbb{R}^k$ evolves in a prescribed set K i.e., under which it holds that for all $t \in [0,T]$ and for all $x \in K$:

$$X_s^{t,x} \in K \quad a.s.\ \omega \in \Omega,\ \forall s \in [t,T].$$

where

- $B = (B^i)_{k\times 1}$, B^i, $i = \overline{1,k}$, are independent fractional Brownian motions with Hurst parameter H, $\frac{1}{2} < H < 1.$, and the integral with respect to B is a pathwise Riemann-Stieltjes integral;

- $X_0 = (X_0^i)_{d\times 1}$ is a d - dimensional random variable defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$;

- $b(\omega,\cdot,\cdot) : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma(\omega,\cdot,\cdot) : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times k}$ are continuous functions.

A general result on the existence and uniqueness of the solution for multidimensional, time dependent, stochastic differential equations driven by a fractional Brownian motion with Hurst parameter $H > 1/2$ has been given by Nualart, Răşcanu in [16] using a techniques of the classical fractional calculus.

The viability - property, has been extensively studied for deterministic differential equations and inclusions,starting with Nagumo's pioneer work in 1943 (see [2] for references). To our knowledge the first work that gives a characterization of the viability property in a stochastic framework was written by Aubin and Da Prato [3] in 1990 (see also [4], [9]). The key point of their work consists in defining a suitable *Bouligand's stochastic tangent cone* which generalizes the cone used in the study of the viability property for deterministic systems. (for other points of view see also [14], [13] and [15]).

Another approach has been developed by Buckdahn, Quincampoix, Rainer, Răşcanu in [5] which present a unified approach for the study of the viability property of SDE, BSDE and PDE by relating the distance to the constraint to some suitable PDE. More precisely, in the case of an SDE (respectively BSDE) they show that the viability property is equivalent to the fact that the square of the distance function is a viscosity super solution (respectively subsolution) of the PDE.

In this paper we will prove a Nagumo type Theorem on viability proprieties of close bounded subsets with respect to a stochastic differential equation driven by fractional Brownian motion, following an approach inspired by the work of Nualart and Răşcanu [16].

The organization of the paper is as follows. In Section 2 we recall some classical definitions and consider the assumptions on the coefficients supposed to hold. In Section 3 we state our main result. Section 4 contains the deterministic more general result and Section 5 contains the proof of the result in the stochastic case.

2 Preliminaries

2.1 Generalized Stieltjes integral

Let $d, k \in \mathbb{N}^*$.Given a matrix $A = \left(a^{i,j}\right)_{d\times k}$ and a vector $y = \left(y^i\right)_{d\times 1}$ we denote $|A|^2 = \sum_{i,j} \left|a^{i,j}\right|^2$ and $|y|^2 = \sum_i \left|y^i\right|^2$.

Let $t \in [0, T]$ be fixed. Denote by $W^{\alpha,\infty}(t, T; \mathbb{R}^d)$, $0 < \alpha < 1$, the space of continuous functions $f : [t, T] \to \mathbb{R}^d$ such that

$$\|f\|_{\alpha,\infty;[t,T]} := \sup_{s\in[t,T]} \left(|f(s)| + \int_t^s \frac{|f(s) - f(r)|}{(s-r)^{\alpha+1}} dr \right) < \infty.$$

A equivalent norm can be defined by

$$|||f|||_{\alpha,\lambda;[t,T]} \overset{def}{=} \sup_{s\in[t,T]} e^{-\lambda s} \left(|f(s)| + \int_t^s \frac{|f(s) - f(r)|}{(s-r)^{\alpha+1}} dr \right)$$

for any $\lambda \geq 0$.
For any $0 < \mu \leq 1$, denote by $C^{\mu}\left([t,T]\,;\mathbb{R}^d\right)$ the space of μ- Holder continuous functions $f : [t,T] \to \mathbb{R}^d$, equipped with the norm

$$\|f\|_{\mu;[t,T]} \overset{def}{=} \|f\|_{\infty;[t,T]} + \sup_{t \leq s < r \leq T} \frac{|f(s) - f(r)|}{(s-r)^{\mu}} < \infty,$$

where $\|f\|_{\infty;[t,T]} := \sup_{s\in[t,T]} |f(s)|$. We have, for all $0 < \varepsilon < \alpha$

$$C^{\alpha+\varepsilon}([t,T]\,;\mathbb{R}^d) \subset W^{\alpha,\infty}(t,T;\mathbb{R}^d) \subset C^{\alpha-\varepsilon}([t,T]\,;\mathbb{R}^d).$$

with continuous embeddings.

Fix a parameter $0 < \alpha < \frac{1}{2}$. Denote by $\tilde{W}^{1-\alpha,\infty}(t,T;\mathbb{R}^d)$ the space of continuous functions $g : [t,T] \to \mathbb{R}^k$ such that

$$\|g\|_{\tilde{W}^{1-\alpha,\infty}(t,T;\mathbb{R}^d)} := |g(t)| + \sup_{t<r<s<T} \left(\frac{|g(s)-g(r)|}{(s-r)^{1-\alpha}} + \int_r^s \frac{|g(y)-g(r)|}{(y-r)^{2-\alpha}} dy \right) < \infty.$$

Clearly,

$$C^{1-\alpha+\varepsilon}([t,T]\,;\mathbb{R}^d) \subset \tilde{W}^{1-\alpha,\infty}(t,T;\mathbb{R}^d) \subset C^{1-\alpha}([t,T]\,;\mathbb{R}^d), \quad \forall \varepsilon > 0.$$

Denoting

$$\Lambda_\alpha(g;[t,T]) \overset{def}{=} \frac{1}{\Gamma(1-\alpha)} \sup_{t<r<s<T} \left| \left(D^{1-\alpha}_{s-} g_{s-}\right)(r) \right|,$$

where

$\Gamma(\alpha) = \int_0^\infty p^{\alpha-1} e^{-p} dp$ is the Euler function and

$$\left(D^{1-\alpha}_{s-} g_{s-}\right)(r) = \frac{e^{i\pi(1-\alpha)}}{\Gamma(\alpha)} \left(\frac{g(r)-g(s)}{(s-r)^{1-\alpha}} + (1-\alpha)\int_r^s \frac{g(r)-g(y)}{(y-r)^{2-\alpha}} dy \right) 1_{(t,s)}(r)$$

we have

$$\Lambda_\alpha(g;[t,T]) \leq \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \|g\|_{\tilde{W}^{1-\alpha,\infty}(t,T;\mathbb{R}^d)}.$$

Note that

$$\Lambda_\alpha(g;[t,T]) \leq \Lambda_\alpha(g;[0,T]) \left(\overset{def}{=} \Lambda_\alpha(g) \right).$$

Let $W^{\alpha,1}(t,T;\mathbb{R}^d)$ the space of measurable functions f on $[t,T]$ such that

$$\|f\|_{\alpha,1;[t,T]} \overset{def}{=} \int_t^T \left[\frac{|f(s)|}{(s-t)^{\alpha}} + \int_t^s \frac{|f(s)-f(y)|}{(s-y)^{\alpha+1}} dy \right] ds < \infty.$$

Clearly $W^{\alpha,\infty}(t,T;\mathbb{R}^d) \subset W^{\alpha,1}(t,T;\mathbb{R}^d)$ and $\|f\|_{\alpha,1;[t,T]} \leq \left(T + \frac{T^{1-\alpha}}{1-\alpha}\right) \|f\|_{\alpha,\infty;[t,T]}$. Denote

$$\left(D^{\alpha}_{t+} f\right)(r) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(r)}{(r-t)^{\alpha}} + \alpha \int_t^r \frac{f(r)-f(y)}{(r-y)^{\alpha+1}} dy \right) 1_{(t,T)}(r). \tag{3}$$

Definition 2 *Let $0 < \alpha < \frac{1}{2}$. If $f \in W^{\alpha,1}(t,T;\mathbb{R}^{dxk})$ and $g \in \tilde{W}^{1-\alpha,\infty}(t,T;\mathbb{R}^k)$, then defining*

$$\int_t^s f(r)\,dg(r) \stackrel{def}{=} (-1)^\alpha \int_t^s \left(D_{t+}^\alpha f\right)(r)\left(D_{s-}^{1-\alpha} g_{s-}\right)(r)\,dr. \tag{4}$$

the integral $\int_t^s f dg$ exists for all $s \in [t,T]$ and

$$\begin{aligned} \left|\int_t^T f(r)\,dg(r)\right| &\leq \sup_{t \leq r < s \leq T} \left|\left(D_{s-}^{1-\alpha} g_{s-}\right)(r)\right| \int_t^T \left|\left(D_{t+}^\alpha f\right)(s)\right|\,ds \\ &\leq \Lambda_\alpha(g;[t,T])\,\|f\|_{\alpha,1;[t,T]}\ . \end{aligned} \tag{5}$$

We give two continuity properties of this integral used in the paper. Let $t \leq s < \tau \leq T$. We denote

$$G_{s,\tau}(f) = \int_s^\tau f(r)\,dg(r) = G_{t,\tau}(f) - G_{t,s}(f)$$

Proposition 3 *Let $0 < \alpha < \dfrac{1}{2}$.*
(a) $G_{t,T} : W^{\alpha,\infty}(t,T;\mathbb{R}^d) \to C^{1-\alpha}([t,T];\mathbb{R}^d)$ *is a linear continuous map and*

$$\|G_{t,\cdot}(f)\|_{1-\alpha;[t,T]} \leq A_{\alpha,T}^{(1)}\ \Lambda_\alpha(g;[t,T])\,\|f\|_{\alpha,\infty;[t,T]} \tag{6}$$

where $A_{\alpha,T}^{(1)}$ is a positive constant depending only on α and T; $A_{\alpha,T}^{(1)} \leq 4+3T$.
(b) $G_{t,T} : W^{\alpha,\infty}(t,T;\mathbb{R}^d) \to W^{\alpha,\infty}(t,T;\mathbb{R}^d)$ *is a linear continuous map and for all $\lambda > 1$*

$$|||G_{t,\cdot}(f)|||_{\alpha,\lambda;[t,T]} \leq \frac{\Lambda_\alpha(g;[t,T])}{\lambda^{1-2\alpha}} A_{\alpha,T}^{(2)}\ \ \|f\|_{\alpha,\lambda;[t,T]}\,, \tag{7}$$

where $A_{\alpha,T}^{(2)}$ is a positive constant depending only on α and T; $A_{\alpha,T}^{(2)} = \dfrac{4}{1-2\alpha}\left(\dfrac{2}{\alpha} + T^\alpha\right)$.

Proof. (a) Let $t \leq \tau < s \leq T$. From the definition of the integral we have

$$|G_{\tau,s}(f)| \leq \Lambda_\alpha(g;[t,T]) \int_\tau^s \left(\frac{|f(\theta)|}{(\theta-\tau)^\alpha} + \alpha\int_\tau^\theta \frac{|f(\theta)-f(u)|}{(\theta-u)^{\alpha+1}}du\right)d\theta \tag{8}$$

and therefore

$$|G_{\tau,s}(f)| \leq \Lambda_\alpha(g;[t,T])\,(2+T^\alpha)\,(s-\tau)^{1-\alpha}\,\|f\|_{\alpha,\infty;[t,T]}\ . \tag{9}$$

Hence the inequality (6) follows with $A_{\alpha,T}^{(1)} = \frac{T^{1-\alpha}}{1-\alpha} + T + 2 + T^\alpha \leq 4+3T$.

(b) We have

$$
\begin{aligned}
&\int_t^s \frac{\left|G_{t,s}(f) - G_{t,r}(f)\right|}{(s-r)^{1+\alpha}} dr \\
&\quad \leq \Lambda_\alpha(g;[t,T]) \int_t^s (s-r)^{-\alpha-1} \times \left(\int_r^s \frac{|f(\theta)|}{(\theta-r)^\alpha} d\theta + \alpha \int_r^\theta \frac{|f(\theta)-f(u)|}{(\theta-u)^{\alpha+1}} du d\theta \right) dr \\
&\quad \leq \Lambda_\alpha(g;[t,T]) \int_t^s |f(\theta)| \int_t^\theta (s-r)^{-\alpha-1} (\theta-r)^{-\alpha} dr d\theta + \\
&\quad + \Lambda_\alpha(g) \alpha \int_t^s \int_t^\theta \frac{|f(\theta)-f(u)|}{(\theta-u)^{\alpha+1}} \left(\int_t^u (s-r)^{-\alpha-1} dr \right) du d\theta.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\int_t^\theta (s-r)^{-\alpha-1} (\theta-r)^{-\alpha} dr \\
&= (s-\theta)^{-2\alpha} \int_0^{\frac{\theta-t}{s-\theta}} (1+u)^{-\alpha-1} u^{-\alpha} du \\
&\leq (s-\theta)^{-2\alpha} \left(\int_0^1 (1+u)^{-\alpha-1} u^{-\alpha} du + \int_1^\infty (1+u)^{-\alpha-1} u^{-\alpha} du \right) \\
&\leq \left(\frac{1}{1-\alpha} + \frac{1}{\alpha} \right) (s-\theta)^{-2\alpha}.
\end{aligned}
$$

and for $t < r < u < \theta < s$

$$
\int_t^u (s-r)^{-\alpha-1} dr \leq \frac{1}{\alpha} (s-u)^{-\alpha} \leq \frac{T^\alpha}{\alpha} (s-t)^{-2\alpha},
$$

then it follows

$$
\begin{aligned}
&\left|G_{t,s}(f)\right| + \int_t^s \frac{\left|G_{t,s}(f) - G_{t,r}(f)\right|}{(s-r)^{1+\alpha}} dr \\
&\quad \leq \Lambda_\alpha(g;[t,T]) \left(\frac{1}{(1-\alpha)\alpha} + T^\alpha \right) \\
&\quad\quad \times \int_t^s \left((s-\theta)^{-2\alpha} + (\theta-t)^{-\alpha} \right) \left(|f(\theta)| + \int_t^\theta \frac{|f(\theta)-f(u)|}{(\theta-u)^{\alpha+1}} du \right) d\theta.
\end{aligned}
$$

The inequality (7) clearly follows since

$$
\begin{aligned}
&\int_t^s e^{-\lambda(s-\theta)} \left[(s-\theta)^{-2\alpha} + (\theta-t)^{-\alpha} \right] d\theta \\
&\quad = \frac{1}{\lambda^{1-2\alpha}} \int_0^{\lambda(s-t)} e^{-u} u^{-2\alpha} du + \frac{1}{\lambda^{1-\alpha}} e^{-\lambda(s-t)} \int_0^{\lambda(s-t)} e^{u} u^{-\alpha} du \\
&\quad \leq \frac{1}{\lambda^{1-2\alpha}} \frac{4}{1-2\alpha}
\end{aligned}
$$

■

2.2 Assumptions and notations

Consider the equation on $\mathbb{R}^d$

$$X_s^i = X_0^i + \int_0^s b^i(r, X_r)dr + \sum_{j=1}^m \int_0^s \sigma^{i,j}(r, X_r)\, dB_r^j, \;\; s \in [0, T], \tag{10}$$

$i = 1, ..., d$, where the processes B^j, $j = 1, ..., k$ are independent fractional Brownian motions with Hurst parameter H defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, X_0 is a d - dimensional random variable, and the coefficients $\sigma^{i,j}, b^i : \Omega \times [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ are measurable functions. Setting $\sigma = (\sigma^{i,j})_{d\times k}$, $b = (b^i)_{d\times 1}$, $B_s = (B_s^j)_{k\times 1}$ and $X_s = (X_s^i)_{d\times 1}$ then we can write the equation (10) in a simpler form

$$X_s = X_0 + \int_0^s b(r, X_r)dr + \int_0^s \sigma(r, X_r)\, dB_r, \;\; s \in [0, T].$$

Let us consider the following assumptions on the coefficients, which are supposed to hold for $\mathbb{P}$ - almost all $\omega \in \Omega$. The constants M_0, M_R, L_0, L_R may depend on ω.

($\mathbf{H}_1$) $\sigma(t, x)$ is differentiable in x, and there exist some constants β, δ, $0 < \beta, \delta \le 1$, and for every $R \ge 0$ there exists $M_R > 0$ such that the following properties hold for all $t \in [0, T]$, $\mathbb{P} - a.s.\ \omega \in \Omega$:

$$(H_\sigma) : \begin{cases} i) & |\sigma(t, x) - \sigma(s, y)| \le M_0 \left(|t - s|^\beta + |x - y|\right), \quad \forall x, y \in \mathbb{R}^d, \\ ii) & |\nabla_x\sigma(t, y) - \nabla_x\sigma(s, z)| \le M_R \left(|t - s|^\beta + |y - z|^\delta\right), \; \forall\, |y|, |z| \le R, \end{cases}$$

where $\nabla_x\sigma(t, x) = (\nabla_x\sigma^{i,j}(t, x))_{i=\overline{1,d},\; j=\overline{1,k}}$ and

$$|\nabla_x\sigma(t, x)|^2 = \sum_{\ell=1}^d \sum_{i=1}^d \sum_{j=1}^k \left|\partial_{x_\ell}\sigma^{i,j}(t, x)\right|^2$$

Remark that for all $x \in \mathbb{R}^d$

$$|\sigma(t, x)| \le |\sigma(0, 0)| + M_0 \left(|t|^\beta + |x|\right) \le M_{0,T}(1 + |x|)$$

where $M_{0,T} = |\sigma(0, 0)| + M_0 + M_0T$.

Let

$$\alpha_0 = \min\left\{\frac{1}{2}, \beta, \frac{\delta}{1 + \delta}\right\}.$$

With respect to the coefficient b we assume

($\mathbf{H}_2$) There exist $\mu \in (1 - \alpha_0, 1]$ and for every $R \ge 0$ there exists $L_R > 0$ such that the following properties hold for all $t \in [0, T]$, $\mathbb{P} - a.s.\ \omega \in \Omega$:

$$(H_b) : \begin{cases} i) & |b(r, x) - b(s, y)| \le L_R \left(|r - s|^\mu + |x - y|\right), \quad \forall\, |x|, |y| \le R, \\ ii) & |b(t, x)| \le L_0(1 + |x|), \; \forall x \in \mathbb{R}^d. \end{cases}$$

From the work of D. Nualart and A. Răşcanu [16] we deduce that under the assumptions (H_1) and (H_2) with $\beta > 1 - H$ and $\delta > \frac{1}{H} - 1$ and for every fixed $(t, \xi) \in [0, T] \times L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d))$, the SDE (10) has a unique solution $X^{t,\xi} \in L^0(\Omega, \mathcal{F}, \mathbb{P}; W^{\alpha,\infty}(t, T; \mathbb{R}^d))$, for all $\alpha \in (1 - H, \alpha_0)$. Moreover, for $\mathbb{P}$ - almost all $\omega \in \Omega$

$$X(\omega, \cdot) = \left(X^i(\omega, \cdot)\right)_{d\times 1} \in C^{1-\alpha}\left(0, T; \mathbb{R}^d\right).$$

In this paper we develop an other proof and we obtain a stronger existence result: if the starting point ξ belongs to a closed set $K \subset \mathbb{R}^d$ then there exists a unique solution $X^{t,\xi}$ and the solution evolves in K.

2.3 Basic estimates

Fix a parameter $0 < \alpha < \dfrac{1}{2} \wedge \beta$. Given two functions $f \in W^{\alpha,1}(t, T; \mathbb{R}^d)$ and $g \in \tilde{W}^{1-\alpha,\infty}(t, T; \mathbb{R}^k)$ we denote

$$G_{t,s}^{(\sigma)}(f) = \int_t^s \sigma(r, f(r))\, dg(r).$$

From Proposition 3 we infer

Corollary 4 *Let the assumption (H_1) be satisfied and $0 < \alpha < \frac{1}{2} \wedge \beta$. Let $f \in W^{\alpha,\infty}\left(t, T; \mathbb{R}^d\right)$ and $g \in \tilde{W}^{1-\alpha,\infty}(t, T; \mathbb{R}^k)$. Then $G_{t,\cdot}^{(\sigma)}(f) \in C^{1-\alpha}\left(t, T; \mathbb{R}^d\right)$ and for all $\lambda \geq 1$*

$$\begin{aligned} &i) \quad \left\| G_{t,\cdot}^{(\sigma)}(f) \right\|_{1-\alpha;[t,T]} \leq C_0^{(\sigma 1)} \Lambda_\alpha(g; [t, T]) \left(1 + \|f\|_{\alpha,\infty;[t,T]}\right), \\ &ii) \quad \left\|\left| G_{t,\cdot}^{(\sigma)}(f) \right|\right\|_{\alpha,\lambda;[t,T]} \leq \frac{C_0^{(\sigma 2)} \Lambda_\alpha(g; [t, T])}{\lambda^{1-2\alpha}} \left(1 + \|f\|_{\alpha,\lambda;[t,T]}\right), \end{aligned} \tag{11}$$

where $C_0^{(\sigma 1)}$ and $C_0^{(\sigma 2)}$ are constants which only depend on $M_{0,T}$, M_0, T, α, β.

Proof. From Proposition 3 we have

$$\left\| G_{t,\cdot}^{(\sigma)}(f) \right\|_{1-\alpha;[t,T]} = \|G_{t,\cdot}(\sigma(\cdot, f(\cdot)))\|_{1-\alpha;[t,T]} \leq A_{\alpha,T}^{(1)}\ \Lambda_\alpha(g; [t, T])\, \|\sigma(\cdot, f(\cdot))\|_{\alpha,\infty;[t,T]}$$

and

$$\left\|\left| G_{t,\cdot}^{(\sigma)}(f) \right|\right\|_{\alpha,\lambda;[t,T]} \leq \frac{\Lambda_\alpha(g; [t, T])}{\lambda^{1-2\alpha}} A_{\alpha,T}^{(2)}\ \|\sigma(\cdot, f(\cdot))\|_{\alpha,\lambda;[t,T]}$$

Now the inequalities (11-i, ii) clearly follow, since for all $\lambda \geq 0$,

$$\begin{aligned}
&\left\|\sigma\left(\cdot, f\left(\cdot\right)\right)\right\|_{\alpha,\lambda;[t,T]} \\
&= \sup_{r\in[t,T]} e^{-\lambda r}\left[\left|\sigma\left(r, f\left(r\right)\right)\right| + \int_t^r \frac{\left|\sigma\left(r, f\left(r\right)\right) - \sigma\left(s, f\left(s\right)\right)\right|}{\left(r-s\right)^{1+\alpha}} ds\right] \\
&\leq \sup_{r\in[t,T]} e^{-\lambda r}\left[M_{0,T}\left(1+\left|f\left(r\right)\right|\right) + M_0 \int_t^r \frac{\left(r-s\right)^{\beta} + \left|f\left(r\right) - f\left(s\right)\right|}{\left(r-s\right)^{1+\alpha}} ds\right] \\
&\leq \left(M_{0,T} + M_0\right)\left(1 + \frac{T^{\beta-\alpha}}{\beta-\alpha}\right)\left(1 + \|f\|_{\alpha,\lambda;[t,T]}\right)
\end{aligned}$$

■

Lemma 5 *Let the assumption (H_1) be satisfied and $0 < \alpha < \frac{1}{2} \wedge \beta$. Let $f, h \in W^{\alpha,\infty}\left(t, T; \mathbb{R}^d\right)$ and $g \in \tilde{W}^{1-\alpha,\infty}(t, T; \mathbb{R}^k)$. Then for $\|f\|_{\infty;[t,T]} \leq R$, $\|h\|_{\infty;[t,T]} \leq R$ it follows*

$$\begin{aligned}
\left\|G_{t,\cdot}^{(\sigma)}\left(f\right) - G_{t,\cdot}^{(\sigma)}\left(h\right)\right\|_{\alpha,\lambda;[t,T]} &\leq \frac{C_R^{(\sigma 3)}\Lambda_\alpha(g;[t,T])}{\lambda^{1-2\alpha}} \\
&\quad\times\left(1 + \Delta_{[t,T]}\left(f\right) + \Delta_{[t,T]}\left(h\right)\right)\|f-h\|_{\alpha,\lambda;[t,T]}
\end{aligned} \tag{12}$$

for all $\lambda \geq 1$, where

$$\Delta_{[t,T]}\left(f\right) = \sup_{r\in[t,T]} \int_t^r \frac{\left|f_r - f_s\right|^{\delta}}{(r-s)^{\alpha+1}}\, ds,$$

and $C_R^{(\sigma 3)}$ is a constant only depending of M_0, M_R, T, α, β.

Remark 6 *If $0 < \alpha < \frac{\delta}{1+\delta}$, $\|f\|_{\infty;[t,T]} \leq R$ and $\|f\|_{1-\alpha;[t,T]} \leq C_R$, with C_R a constant independent of t, then*

$$\Delta_{[t,T]}\left(f\right) \leq C_R \sup_{r\in[t,T]} \int_t^r \frac{\left(r-s\right)^{(1-\alpha)\delta}}{(r-s)^{\alpha+1}} ds \leq \frac{T^{\delta-\alpha(1+\delta)}}{\delta - \alpha\left(1+\delta\right)} C_R\ .$$

Proof of Lemma 5. For the proof we use the ideas from [16]. By the inequality (7) we have

$$\left\|G_{t,\cdot}^{(\sigma)}\left(f\right) - G_{t,\cdot}^{(\sigma)}\left(h\right)\right\|_{\alpha,\lambda;[t,T]} \leq \frac{\Lambda_\alpha(g;[t,T])A_{\alpha,T}^{(2)}}{\lambda^{1-2\alpha}}\left\|\sigma\left(\cdot, f\left(\cdot\right)\right) - \sigma\left(\cdot, h\left(\cdot\right)\right)\right\|_{\alpha,\lambda;[t,T]}. \tag{13}$$

Remark that if $\left|x\right|, \left|y\right|, \left|u\right|, \left|v\right| \leq R$, then

$$\begin{aligned}
&\left|\sigma^{i,j}\left(r, x\right) - \sigma^{i,j}\left(r, u\right) - \sigma^{i,j}\left(s, y\right) + \sigma^{i,j}\left(s, v\right)\right| \\
&\quad= \left|\int_0^1 \langle x-u, \nabla_x\sigma^{i,j}(r, \theta x + (1-\theta)u\rangle\, d\theta - \int_0^1 \langle y-v, \nabla_x\sigma^{i,j}(s, \theta y + (1-\theta)v)\rangle\, d\theta\right| \\
&\quad\leq \left|\int_0^1 \langle x-y-u+v, \nabla_x\sigma^{i,j}(s, \theta y + (1-\theta)v)\rangle\, d\theta\right| \\
&\qquad+ \left|\int_0^1 \langle x-u, \nabla_x\sigma^{i,j}(r, \theta x + (1-\theta)u) - \nabla_x\sigma^{i,j}(s, \theta y + (1-\theta)v)\rangle\, d\theta\right|
\end{aligned}$$

and therefore

$$\begin{aligned}\left|\sigma\left(r,x\right)-\sigma\left(r,u\right)-\sigma\left(s,y\right)+\sigma\left(s,v\right)\right| &\le M_0\left|x-y-u+v\right| \\ &\quad + M_R\left|x-u\right|\left(\left|s-r\right|^{\beta}+\left|x-y\right|^{\delta}+\left|u-v\right|^{\delta}\right).\end{aligned}$$

Hence for $\|f\|_{\infty,[t,T]}\le R$ and $\|h\|_{\infty,[t,T]}\le R$

$$\begin{aligned}&\left\|\sigma\left(\cdot,f\left(\cdot\right)\right)-\sigma\left(\cdot,h\left(\cdot\right)\right)\right\|_{\alpha,\lambda;[t,T]}\\ &\quad=\sup_{r\in[t,T]}e^{-\lambda r}\Bigg[\left|\sigma\left(r,f\left(r\right)\right)-\sigma\left(r,h\left(r\right)\right)\right|\\ &\qquad\qquad+\int_t^r\frac{\left|\sigma\left(r,f\left(r\right)\right)-\sigma\left(r,h\left(r\right)\right)-\sigma\left(s,f\left(s\right)\right)+\sigma\left(s,h\left(s\right)\right)\right|}{\left(r-s\right)^{1+\alpha}}ds\\ &\quad\le\left(M_0+M_R\right)\sup_{r\in[t,T]}e^{-\lambda r}\Bigg[\left|f\left(r\right)-h\left(r\right)\right|+\int_t^r\frac{\left|f\left(r\right)-f\left(s\right)-h\left(r\right)+h\left(s\right)\right|}{\left(r-s\right)^{1+\alpha}}ds\\ &\qquad\qquad+\left|f\left(r\right)-h\left(r\right)\right|\left(\frac{1}{\beta-\alpha}\left(r-t\right)^{\beta-\alpha}+\Delta_{[t,T]}\left(f\right)+\Delta_{[t,T]}\left(h\right)\right)\Bigg],\end{aligned}$$

where

$$\Delta_{[t,T]}\left(f\right)=\sup_{r\in[t,T]}\int_t^r\frac{\left|f\left(r\right)-f\left(s\right)\right|^{\delta}}{\left(r-s\right)^{1+\alpha}}$$

and similar for h. We conclude that

$$\begin{aligned}&\left\|\sigma\left(\cdot,f\left(\cdot\right)\right)-\sigma\left(\cdot,h\left(\cdot\right)\right)\right\|_{\alpha,\lambda;[t,T]}\\ &\le\left(M_0+M_R\right)\left(1+\frac{T^{\beta-\alpha}}{\beta-\alpha}\right)\left[1+\Delta_{[t,T]}\left(f+\Delta_{[t,T]}\left(h\right)\right)\right]\left\|f-h\right\|_{\alpha,\lambda;[t,T]}\end{aligned}$$

and the inequality (12) now follows from (13).

■

We also give similar estimates for

$$F_{t,s}^{(b)}\left(f\right)=\int_t^s b\left(r,f(r)\right)dr.$$

where b satisfies the assumptions (H_2). Very similar estimates are given in the paper of Nualart&Răşcanu [16].

Lemma 7 *Let $f\in W^{\alpha,\infty}(t,T;\mathbb{R}^d)$. Then $F_{t,\cdot}^{(b)}\left(f\right)=\int_t^{\cdot}b\left(y,f(y)\right)dy\in C^{1-\alpha}\left(t,T;\mathbb{R}^d\right)$ and for all $\lambda\ge1$:*

$$\begin{aligned}&(j)\quad\left\|F_{t,\cdot}^{(b)}\left(f\right)\right\|_{1-\alpha;[t,T]}\le C_0^{(b1)}\left(1+\|f\|_{\infty;[t,T]}\right),\\ &(jj)\quad\left\|F_{t,\cdot}^{(b)}\left(f\right)\right\|_{\alpha,\lambda;[t,T]}\le\frac{C_0^{(b2)}}{\lambda^{\alpha}}\left(1+\|f\|_{\alpha,\lambda;[t,T]}\right),\end{aligned}\tag{14}$$

where $C_0^{(b1)}$ and $C_0^{(b2)}$ are positive constants depending only on α, T and L_0 .
If $f, h \in W^{\alpha,\infty}(t,T;\mathbb{R}^d)$ such that $\|f\|_{\infty;[t,T]} \leq R$, $\|h\|_{\infty;[t,T]} \leq R$, then

$$\left\| F_{t,\cdot}^{(b)}(f) - F_{t,\cdot}^{(b)}(h) \right\|_{\alpha,\lambda;[t,T]} \leq \frac{C_R^{(b3)}}{\lambda^\alpha} \|f-h\|_{\alpha,\lambda;[t,T]} \tag{15}$$

for all $\lambda \geq 1$, where $C_R^{(b3)}$ constants depending only on α, T and L_R from (H_2).

Proof. It is easy to see that $F_{t,\cdot}^{(b)}(f) \in C^1([t,T])$ and for $t \leq r \leq s \leq T$

$$\left| F_{t,s}^{(b)}(f) - F_{t,r}^{(b)}(f) \right| = \left| F_{r,s}^{(b)}(f) \right| \leq L_0 \left(1 + \|f\|_{\infty;[t,T]} \right)(s-r).$$

Hence the inequality (14-j) follows with $C_0^{(b1)} = L_0(T + T^\alpha)$.

Denoting

$$F_{t,s}(f) = \int_t^s f(r)\, dr$$

we remark that

$$\begin{aligned}
&|F_{t,s}(f)| + \int_t^s \frac{|F_{t,s}(f) - F_{t,r}(f)|}{(s-r)^{\alpha+1}}\, dr \\
&\qquad \leq \int_t^s |f(r)|\, dr + \int_t^s (s-r)^{-\alpha-1} \left(\int_r^s |f(u)|\, du \right) dr \\
&\qquad \leq \int_t^s |f(r)|\, dr + \frac{1}{\alpha} \int_t^s (s-t)^{-\alpha} \quad |f(u)|\, du \\
&\qquad \leq \left(T^\alpha + \frac{1}{\alpha} \right)(s-t)^{-\alpha} \int_t^s |f(r)|\, dr,
\end{aligned}$$

and therefore

$$\begin{aligned}
\left\| F_{t,\cdot}^{(b)}(f) \right\|_{\alpha,\lambda;[t,T]} &\leq \left(T^\alpha + \frac{1}{\alpha} \right) \sup_{s\in[t,T]} e^{-\lambda s}(s-t)^{-\alpha} \int_t^s |b(r, f(r))|\, dr \\
&\leq L_0 \left(T^\alpha + \frac{1}{\alpha} \right) \sup_{s\in[t,T]} \int_t^s e^{-\lambda(s-r)} \frac{1 + e^{-\lambda r}|f(r))|}{(s-r)^\alpha} dr, \\
&\leq L_0 \left(T^\alpha + \frac{1}{\alpha} \right) \frac{1}{\lambda^\alpha} \frac{T^{1-2\alpha}}{1-2\alpha} \left(1 + \|f\|_{\alpha,\lambda;[t,T]} \right)
\end{aligned}$$

that clearly yields (14-jj) with $C_0^{(b2)} = L_0 \left(T^\alpha + \frac{1}{\alpha} \right) \frac{T^{1-2\alpha}}{1-2\alpha}$, since $e^{-\lambda(s-r)}(s-r)^{-a} \leq \frac{1}{\lambda^a}(s-r)^{-2a}$.

Let now $f, h \in W^{\alpha,\infty}(t,T;\mathbb{R}^d)$ and $|f| \leq R$ and $|h| \leq R$. Then as here above, for all $\lambda \geq 1$,

$$\begin{aligned}
\left\| F_{t,\cdot}^{(b)}(f) - F_{t,\cdot}^{(b)}(h) \right\|_{\alpha,\lambda;[t,T]} &\leq \left(T^\alpha + \frac{1}{\alpha} \right) \sup_{s\in[t,T]} e^{-\lambda s}(s-t)^{-\alpha} \int_t^s |b(r, f(r)) - b(r, h(r))|\, dr \\
&\leq L_R \left(T^\alpha + \frac{1}{\alpha} \right) \frac{1}{\lambda^\alpha} \frac{T^{1-2\alpha}}{1-2\alpha} \ \|f-h\|_{\alpha,\lambda;[t,T]} \ .
\end{aligned}$$

Hence the inequality (15)holds with $C_R^{(b3)} = L_R \left(T^\alpha + \dfrac{1}{\alpha} \right) \dfrac{T^{1-2\alpha}}{1-2\alpha}$.

■

Finally we present some auxiliary estimates used in the sequel.

Lemma 8 *Let the assumptions* (H_1) *and* (H_2) *be satisfied and* $0 < \alpha < \beta \wedge \frac{1}{2}$. *If* Y *is a Holder continuous function with* $\|Y\|_{1-\alpha;[t,T]} \leq R$ *then there exist some positive constants* $C_R^{(1)} = (R+1+T) L_R$, $C_R^{(2)} = C^{(2)}(R, M_0, T, \alpha, \beta, \Lambda_\alpha(g))$, $C_R^{(3)} = 2(1+R) L_0$ *and* $C_R^{(4)} = C^{(4)}(R, M_0, T, \alpha, \beta, \Lambda_\alpha(g))$ *such that for all* $0 \leq t \leq s \leq T$:

$$\begin{aligned} &(a) \quad \left| \int_t^s [b(r, Y_r) - b(t, Y_t)]\, dr \right| \leq C_R^{(1)} (s-t)^{2-\alpha} \quad \text{and} \\ &(b) \quad \left| \int_t^s [\sigma(r, Y_r) - \sigma(t, Y_t)]\, dg(r) \right| \leq C_R^{(2)} (s-t)^{1+\min\{\beta-\alpha, 1-2\alpha\}}. \end{aligned} \tag{16}$$

and for all $0 \leq t \leq \tau \leq s \leq T$:

$$\begin{aligned} &(c) \quad \left| \int_\tau^s [b(r, Y_r) - b(t, Y_t)]\, dr \right| \leq C_R^{(3)} (s-\tau) \quad \text{and} \\ &(d) \quad \left| \int_\tau^s [\sigma(r, Y_r) - \sigma(t, Y_t)]\, dg(r) \right| \leq C_R^{(4)} (s-\tau)^{1-\alpha}. \end{aligned} \tag{17}$$

Proof. (a) We have

$$\begin{aligned} \left| \int_t^s [b(r, Y_r) - b(t, Y_t)]\, dr \right| &\leq (s-t) \sup_{r \in [t,s]} |b(r, Y_r) - b(t, Y_t)| \\ &\leq (s-t) \sup_{r \in [t,s]} L_R \left(|Y_r - Y_t| + |r-t|^\mu \right) \\ &\leq (s-t) L_R \sup_{r \in [t,s]} \left| [R + (1+T)] (r-t)^{1-\alpha} \right| \\ &\leq C_R^{(1)} (s-t)^{2-\alpha} \end{aligned}$$

(b) By the assumptions (H_1) and Lemma 13 we have

$$\begin{aligned} |\sigma(r, Y_r) - \sigma(\theta, Y_\theta)| &\leq M_0 \left[|r-\theta|^\beta + |Y_r - Y_\theta| \right] \\ &\leq M_0 \left[|r-\theta|^\beta + R |r-\theta|^{1-\alpha} \right] \end{aligned}$$

and

$$\begin{aligned} &\left| \int_t^s [\sigma(r, Y_r) - \sigma(t, Y_t)]\, dg(r) \right| \\ &\qquad \leq \left| \Lambda_\alpha(g) \left(\int_t^s \frac{\sigma(r, Y_r) - \sigma(t, Y_t)}{(r-t)^\alpha} dr + \int_t^s \int_t^r \frac{\sigma(r, Y_r) - \sigma(\theta, Y_\theta)}{(r-\theta)^{1+\alpha}} d\theta dr \right) \right| \end{aligned}$$

$$
\begin{aligned}
&\leq \Lambda_\alpha(g)\, M_0 \left| \int_t^s \left[(r-t)^{\beta-\alpha} + R\,(r-t)^{1-2\alpha} \right] dr \right. \\
&\qquad \left. + \int_t^s \int_t^r \left[(r-\theta)^{\beta-1-\alpha} + R\,(r-\theta)^{-2\alpha} \right] d\theta dr \right| \\
&\leq \tilde{C}_R^{(2)} \left\{ (s-t)^{\beta-\alpha+1} + (s-t)^{2-2\alpha} + (s-t)^{\beta-\alpha+1} + (s-t)^{2-2\alpha} \right\} \\
&\leq C_R^{(2)}\,(s-t)^{1+\min\{\beta-\alpha, 1-2\alpha\}}.
\end{aligned}
$$

(c) We have, by $(H_2 - ii)$,

$$
\begin{aligned}
\left| \int_\tau^s \left[b(r, Y_r) - b(t, Y_t) \right] dr \right| &\leq (s-\tau) \sup_{r\in[\tau,s]} \left| b(r,Y_r) - b(t,Y_t) \right| \\
&\leq L_0\,(s-\tau)\,(2 + |Y_r| + |Y_t|) \\
&\leq 2\,(1+R)\,L_0\,(s-\tau)
\end{aligned}
$$

(d) Using (8) we deduce

$$
\begin{aligned}
&\left| \int_\tau^s \left[\sigma(r,Y_r) - \sigma(t,Y_t) \right] dg(r) \right| \\
&\leq \Lambda_\alpha(g) \left| \int_\tau^s \frac{\sigma(r,Y_r) - \sigma(t,Y_t)}{(r-\tau)^\alpha} dr + \int_\tau^s \int_\tau^r \frac{\sigma(r,Y_r) - \sigma(\theta,Y_\theta)}{(r-\theta)^{1+\alpha}} d\theta dr \right| \\
&\leq \Lambda_\alpha(g)\, M_0 \int_\tau^s \left(\frac{(r-t)^\beta}{(r-\tau)^\alpha} + R \frac{(r-t)^{1-\alpha}}{(r-\tau)^\alpha} \right) dr \\
&\qquad + \Lambda_\alpha(g)\, M_0 \int_\tau^s \int_\tau^r \left((r-\theta)^{\beta-\alpha-1} + R\,(r-\theta)^{-2\alpha} \right) d\theta dr \\
&\leq \tilde{C}_R^{(4)} \left[T^\beta (s-\tau)^{1-\alpha} + T^{1-\alpha}(s-\tau)^{1-\alpha} + (s-\tau)^{\beta+1-\alpha} + (s-\tau)^{2-2\alpha} \right] \\
&\leq C_R^{(4)}\,(s-\tau)^{1-\alpha}.
\end{aligned}
$$

■

3 Fractional viability. Main result.

Consider the stochastic differential equation driven by the fractional Brownian motion B with Hurst parameter $1/2 < H \leq 1$, $\mathbb{P} - a.s.\ \omega \in \Omega$,

$$
X_s^{t,x} = x + \int_t^{s\vee t} b(r, X_r^{t,x}) dr + \int_t^{s\vee t} \sigma(r, X_r^{t,x}) dB_r^H, \quad s \in [t,T], \tag{18}
$$

where

- $B = (B^i)_{k\times 1}$, B^i, $i = \overline{1,k}$, are independent fractional Brownian motions with Hurst parameter H, $\dfrac{1}{2} < H < 1$, and the integral with respect to B is a pathwise Riemann-Stieltjes integral;

- $X_0 = (X_0^i)_{d\times 1}$ is a d - dimensional random variable defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$;

- $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times k}$ are continuous functions.

Definition 9 *Let $\mathcal{K} = \{K(t) : t \in [0,T]\}$ be a family of subsets of $\mathbb{R}^d$. We shall say that $\mathcal{K}$ is viable for the equation (18) if, starting at any time $t \in [0,T]$ and from any point $x \in K(t)$, at least one its solution $X_s^{t,x} \in K(s)$ for all $s \in [t,T]$.*

Definition 10 *The family $\mathcal{K}$ is said to be invariant for the equation (18) if, for any $t \in [0,T]$ and for any starting point $x \in K(t)$, all solutions $\{X_s^{t,x} : s \in [t,T]\}$ of the fractional stochastic differential equation (18) have the property*

$$X_s^{t,x} \in K(s) \text{ for all } s \in [t,T].$$

Remark that, in the case when the equation has a unique solution (which is the case for the equation (18) under the assumptions (H_1) and (H_2)), viability is equivalent with invariance.

Assume that the mappings b and σ from the equations (18) are satisfying (H_1) and (H_2)

Definition 11 *Let $t \in [0,T]$ and $x \in K(t)$. Let $\frac{1}{2} < 1-\alpha < H$. We say that the pair $(b(t,x), \sigma(t,x))$ is $(1-\alpha)-$fractional B^H- contingent to $K(t)$ in (t,x) if there exist random variable $\bar{h} = \bar{h}^{t,x} > 0$, a stochastic process $Q = Q^{t,x} : \Omega \times \left[t, t+\bar{h}\right] \to \mathbb{R}^d$ and for every $R > 0$ such that $|x| \le R$ there exist two random variables $H_R, \tilde{H}_R > 0$ and a constant $\gamma = \gamma_R \in (0,1)$ which are independent of $\left(t, \bar{h}\right)$ (the constants H_R, $\tilde{H}_R$, γ_R depend only on R, L_R, $M_{0,T}$, M_0, L_0, T, α, β, $\Lambda_\alpha\left(B^H\right)$) such that for all $s, \tau \in \left[t, t+\bar{h}\right]$*

$$|Q(s) - Q(\tau)| \le H_R |s-\tau|^{1-\alpha} \quad \text{and} \quad |Q(s)| \le \tilde{H}_R |s-t|^{1+\gamma}$$

satisfying

$$x + (s-t)\, b(t,x) + \sigma(t,x)\left[B_s^H - B_t^H\right] + Q(s) \in K(s).$$

The main result (existence result and characterization of the viability) of our paper is the following

Theorem 12 *Let $\mathcal{K} = \{K(t) : t \in [0,T]\}$ be a family of nonempty closed subsets of $\mathbb{R}^d$. Assume that the maps b and σ from the equations (18) are satisfying (H_1), (H_2) with $\frac{1}{2} < H < 1$, $1 - H < \beta$, $\delta > \frac{1-H}{H}$.*
Let $1 - H < \alpha < \alpha_0$. Then the following assertions are equivalent:

- *$\mathcal{K}$ is viable for the fractional stochastic differential equation (18), i.e. for all $t \in [0,T]$ and for all $x \in K(t)$ there exists a solution $X^{t,x}(\omega, \cdot) \in C^{1-\alpha}\left([t,T];\ \mathbb{R}^d\right)$ of the equation*

$$X_s^{t,x} = x + \int_t^{s\vee t} b(r, X_r^{t,x})dr + \int_t^{s\vee t} \sigma(r, X_r^{t,x})dB_r^H, \quad s \in [t,T],\ a.s.\ \omega \in \Omega,$$

  and $X_s^{t,x} \in K(s)$, for all $s \in [t,T]$.

- *For all $t \in [0,T]$ and all $x \in K(t)$, $(b(t,x), \sigma(t,x))$ is $(1-\alpha)-$fractional B^H-contingent to $K(t)$ in (t,x).*

4 Deterministic approach

Let arbitrary fixed $(t,x) \in [0,T] \times \mathbb{R}^d$. Consider the deterministic differential equation on $\mathbb{R}^d$:

$$X_s^{tx} = x + \int_t^s b(r, X_r^{tx})dr + \int_t^s \sigma\left(r, X_r^{tx}\right) dg(r), \quad s \in [t,T], \tag{19}$$

where $g \in \tilde{W}^{1-\alpha,\infty}(t,T;\mathbb{R}^k)$ and the coefficients $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times k}$ are continuous functions satisfying the assumptions (H_1), (H_2). Let α be arbitrary fixed such that

$$0 < \alpha < \alpha_0 = \left\{\frac{1}{2}, \beta, \frac{\delta}{1+\delta}\right\}.$$

Nualart and Rascanu proved in [16] that if the assumptions (H_1) and (H_2) are satisfied then the equation (19) has a unique solution which is $(1-\alpha)-$ Holder continuous. In the following Lemma we shall proof that the Holder constant of this solution has the form $C_0(1+|x|)$, with C_0 a positive constant depending only on $M_{0,T}$, M_0, L_0 T, α, β.

Lemma 13 *Let the assumptions (H_1) and (H_2) be satisfied. If $X^{t,x}$ is a solution of the equation (19) then $X^{t,x}$ is $(1-\alpha)$-Holder continuous and*

$$\left\|X_\cdot^{t,x}\right\|_{1-\alpha;[t,T]} \leq C_0(1+|x|)$$

where C_0 is a constant depending only on $M_{0,T}$, M_0, L_0 T, α, β, $\Lambda_\alpha(g)$.

Proof. By Corollary 4 and Lemma 7 we have for all $\lambda \geq 1$

$$
\begin{aligned}
&\left\|X_{\cdot}^{t,x}\right\|_{\alpha,\lambda;[t,T]} \\
&\leq |x| + \left\|F_{t,\cdot}^{(b)}\left(X^{t,x}\right)\right\|_{\alpha,\lambda;[t,T]} + \left\|G_{t,\cdot}^{(\sigma)}\left(X^{t,x}\right)\right\|_{\alpha,\lambda;[t,T]} \\
&\leq |x| + \frac{C_0^{(b2)}}{\lambda^{1-2\alpha}}\left(1+\left\|X_{\cdot}^{t,x}\right\|_{\alpha,\lambda;[t,T]}\right) + \frac{\Lambda_a\left(g;[0,T]\right)C_0^{(\sigma 2)}}{\lambda^{1-2\alpha}}\left(1+\left\|X_{\cdot}^{t,x}\right\|_{\alpha,\lambda;[t,T]}\right) \\
&\leq |x| + \frac{1}{2}\left(1+\left\|X^{t,x}\right\|_{\alpha,\lambda;[t,T]}\right)
\end{aligned}
$$

for $\lambda = \lambda_0 \geq 1$ sufficiently large,

$$
\frac{C_0^{(b2)} + \Lambda_a(g)\, C_0^{(\sigma 2)}}{\lambda_0^{1-2\alpha}} \leq \frac{1}{2},
$$

(remark that the constant $C_0^{(b2)} + \Lambda_\alpha(g)\, C_0^{(\sigma 2)}$ is independent of λ). Then we have

$$
\left\|X^{t,x}\right\|_{\alpha,\lambda;[t,T]} \leq 2\left(1+|x|\right) \quad \text{and} \quad \left\|X_{\cdot}^{t,x}\right\|_{\alpha,\infty;[t,T]} \leq 2\left(1+|x|\right)e^{\lambda_0 T}.
$$

On the other hand, using the same lemmas we get

$$
\begin{aligned}
\left\|X^{t,x}\right\|_{1-\alpha;[t,T]} &\leq |x| + \left\|F_{t,\cdot}^{(b)}\left(X^{t,x}\right)\right\|_{1-\alpha;[t,T]} + \left\|G_{t,\cdot}^{(\sigma)}\left(X^{t,x}\right)\right\|_{1-\alpha;[t,T]} \\
&\leq |x| + C_0^{(b1)}\left(1+\left\|X^{t,x}\right\|_{\infty;[t,T]}\right) + \Lambda_\alpha(g)\, C_0^{(\sigma 1)}\left(1+\left\|X^{t,x}\right\|_{\alpha,\infty;[t,T]}\right) \\
&\leq |x| + \left(C_0^{(b1)} + \Lambda_\alpha(g)\, C_0^{(\sigma 1)}\right)\left(1+\left\|X^{t,x}\right\|_{\alpha,\infty;[t,T]}\right) \\
&\leq |x| + \left(C_0^{(b1)} + \Lambda_\alpha(g)\, C_0^{(\sigma 1)}\right)\left(1+2e^{\lambda_0 T} + 2|x|\, e^{\lambda_0 T}\right) \\
&\leq C_0\left(1+|x|\right)
\end{aligned}
$$

Hence $X_{\cdot}^{t,x}$ is $(1-\alpha)-$Holder continuous with the Holder constant $C_0(1+|x|)$ where C_0 depends only on $M_{0,T}$, M_0, L_0 T, α, β and $\Lambda_\alpha(g)$. The proof is now complete.

■

Assume that the maps b and σ from the equations (19) are satisfying (H_1) and (H_2).

Definition 14 (Tangency property) *Let $t \in [0,T]$ and $x \in K(t)$. We say that the pair $(b(t,x), \sigma(t,x))$ is $(1-\alpha)-$fractional $g-$ tangent to $K(t)$ in (t,x) if there exist $\bar{h} = \bar{h}^{t,x} > 0$, and two functions $U = U^{t,x} : \left[t, t+\bar{h}\right] \to \mathbb{R}^d$, $U(t) = 0$, and $V = V^{t,x} : \left[t, t+\bar{h}\right] \to \mathbb{R}^{d\times k}$, $V(t) = 0$, and for every $R > 0$ such that $|x| \leq R$ there exist two constants $D_R, \tilde{D}_R > 0$ independent of $\left(t, \bar{h}\right)$ such that for all $s, \tau \in \left[t, t+\bar{h}\right]$*

$$
|U(\tau) - U(s)| \leq D_R\, |\tau - s|^{1-\alpha} \quad \text{and} \quad |V(\tau) - V(s)| \leq \tilde{D}_R\, |\tau - s|^{\min\{\beta,\ 1-\alpha\}} \tag{20}
$$

and satisfying

$$x+\int_t^s [b(t,x)+U(r)]\,dr+\int_t^s [\sigma(t,x)+V(r)]\,dg(r)\in K(s), \quad \text{for all } s\in\left[t,t+\bar{h}\right].$$

Definition 15 (Contingency property) *Let $t\in[0,T]$ and $x\in K(t)$. We say that the pair $(b(t,x),\sigma(t,x))$ is $(1-\alpha)-$fractional $g-$ contingent to $K(t)$ in (t,x) if there exist $\bar{h}=\bar{h}^{t,x}>0$, a function $Q=Q^{t,x}:\left[t,t+\bar{h}\right]\to\mathbb{R}^d$ and for every $R>0$ such that $|x|\leq R$ there exist two constants G_R,$\tilde{G}_R>0$ independent of $\left(t,\bar{h}\right)$ and a constant $\gamma=\gamma_R\in(0,1)$ also independent of $\left(t,\bar{h}\right)$ (the constants G_R,$\tilde{G}_R$, γ_R depend only on $R, L_R, M_{0,T}$, M_0, L_0 T, α, β and $\Lambda_\alpha(g)$) such that for all $s,\tau\in\left[t,t+\bar{h}\right]$*

$$|Q(\tau)-Q(s)|\leq G_R\,|\tau-s|^{1-\alpha} \quad \text{and} \quad |Q(s)|\leq \tilde{G}_R\,|s-t|^{1+\gamma}$$

and satisfying

$$x+(s-t)\,b(t,x)+\sigma(t,x)\,[g(s)-g(t)]+Q(s)\in K(s), \quad \text{for all } s\in\left[t,t+\bar{h}\right].$$

We can now state the main result of the section.

Theorem 16 *Let $\mathcal{K}=\{K(t):t\in[0,T]\}$ be a family of nonempty closed subsets of $\mathbb{R}^d$. Assume (H_1) and (H_2) are satisfied and*

$$1-\mu<\alpha<\alpha_0=\min\left\{\frac{1}{2},\beta,\frac{\delta}{1+\delta}\right\}.$$

Then the following assertions are equivalent:

(j) $\mathcal{K}$ is $C^{1-\alpha}-$viable for the fractional differential equation (19), i.e. for any $t\in[0,T]$ and for any starting point $x\in K(t)$, there exists a solution $X^{t,x}(\cdot)\in C^{1-\alpha}\left([t,T];\ \mathbb{R}^d\right)$ of the equation

$$X_s^{t,x}=x+\int_t^{s\vee t} b(r,X_r^{t,x})dr+\int_t^{s\vee t}\sigma(r,X_r^{t,x})dg(r),\ \ s\in[t,T], \tag{21}$$

such that $X_s^{t,x}\in K(s)$, for all $s\in[t,T]$.

(jj) For all $t\in[0,T]$ and all $x\in K(t)$ the pair $(b(t,x),\sigma(t,x))$ is $(1-\alpha)-$fractional $g-$ tangent to $K(t)$ in (t,x).

(jjj) For all $t\in[0,T]$ and all $x\in K(t)$ the pair $(b(t,x),\sigma(t,x))$ is $(1-\alpha)-$fractional $g-$ contingent to $K(t)$ in (t,x).

Proof. Let $0 < \varepsilon \leq 1$. We denote by C_R, $C_R^{(1)}$, $C_R^{(2)}$, ... denote a generic positive constant independent of ε and depending only on $R, L_R, M_{0,T}$, M_0, L_0 T, α, β and $\Lambda_\alpha(g)$.

(j) $\Rightarrow$ **(jj)** :

Let $t \in [0,T]$ and $x \in K(t)$ be arbitrary fixed and $X^{t,x} \in C^{1-\alpha}\left([t,T];\, \mathbb{R}^d\right)$ a solution of the equation (21) such that $X_s^{t,x} \in K(s)$, for all $s \in [t,T]$. Let $R_0 > 0$ such that $|x| \leq R_0$. Then by Lemma 13

$$\left\|X^{t,x}\right\|_{1-\alpha;[t,T]} \leq R = C_0(1+R_0)$$

with C_0 depending only on $M_{0,T}$, M_0, L_0 T, α, β and $\Lambda_\alpha(g)$. Let $\bar{h} = \min\{T-t, 1\}$. Then

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dg(r) \in K(s), \quad \forall\, s \in \left[t, t+\bar{h}\right].$$

We clearly have for all $s \in \left[t, t+\bar{h}\right]$

$$X_s^{t,x} = x + \int_t^s \left[b(t,x) + U(r)\right]dr + \int_t^s \left[\sigma(t,x) + V(r)\right]dg(r)$$

where

$$U(r) = b(r, X_r^{t,x}) - b(t,x) \quad \text{and} \quad V(r) = \sigma(r, X_r^{t,x}) - \sigma(t,x).$$

Clearly U and V satisfy (20).

(jj) $\Rightarrow$ **(jjj)** :

Let $|x| \leq R$. We verify that

$$Q(s) = \int_t^s U(r)\,dr + \int_t^s V(r)\,dg(r)$$

satisfies the Holder conditions from the definition of the contingency property. Indeed by we have

$$\left|\int_t^s U(r)\,dr\right| = \left|\int_t^s \left[U(r) - U(t)\right]dr\right| \leq D_R\ |s-t|^{2-\alpha}$$

and

$$\begin{aligned}
\left|\int_t^s V(\theta)\,dg(\theta)\right| &\leq \left|\int_t^s \left[V(\theta) - V(t)\right]dg(\theta)\right| \\
&\leq \Lambda_\alpha(g) \int_t^s \left(\frac{|V(\theta) - V(t)|}{(\theta-t)^\alpha} + \alpha\int_t^\theta \frac{|V(\theta) - V(u)|}{(\theta-u)^{\alpha+1}}du\right)d\theta \\
&\leq C_R^{(1)}(s-t)^{1+\min\{\beta-\alpha, 1-2\alpha\}}
\end{aligned}$$

Hence
$$\left|Q\left(s\right)\right| \leq C_R^{(2)}\left(s-t\right)^{1+\min\{\beta-\alpha,1-2\alpha\}}.$$
Now we prove that Q is $(1-\alpha)$ −Holder continuous on $\left[t,t+\bar{h}\right]$. Let $t\leq\tau\leq s\leq t+\bar{h}$. We have

$$\begin{aligned}\left|\int_\tau^s V\left(\theta\right)dg\left(\theta\right)\right| &\leq \left|\int_\tau^s \left[V\left(\theta\right)-V\left(t\right)\right]dg\left(\theta\right)\right| \\ &\leq \Lambda_\alpha(g)\int_\tau^s\left(\frac{\left|V\left(\theta\right)-V\left(t\right)\right|}{(\theta-\tau)^\alpha}+\alpha\int_\tau^\theta\frac{|V(\theta)-V(u)|}{(\theta-u)^{\alpha+1}}du\right)d\theta \\ &\leq C_R^{(3)}\left(s-\tau\right)^{1-\alpha}\end{aligned}$$

and therefore

$$\begin{aligned}\left|Q\left(s\right)-Q\left(\tau\right)\right| &\leq \left|\int_\tau^s U\left(r\right)dr\right|+\left|\int_\tau^s V\left(\theta\right)dg\left(\theta\right)\right| \\ &\leq C_R^{(4)}\left(s-\tau\right)^{1-\alpha}.\end{aligned}$$

(**jjj**) $\Rightarrow$ (**j**) :

Let us fix $t\in[0,T]$, $x\in K\left(t\right)$ and $0<\varepsilon\leq1$. Let $R_0>0$ be such that $|x|\leq R_0$.

We denote by $\mathcal{A}_\varepsilon\left(t,x\right)$ the set of pairs $(T_X,\ X)$ where $T_X\in[0,T]$ and $X:[t,T_X]\to\mathbb{R}^d$ is a Holder continuous function satisfying

(1) $X_t=x$, $X_s\in K\left(s\right)$ for all $s\in\left[t,T_X\right]$, and there exists a positive constant $B_0\geq R_0$ depending only onR_0, L_{R_0}, $M_{0,T}$, M_0, L_0 T, α, β and $\Lambda_\alpha\left(g\right)$, such that

$$\left\|X\right\|_{1-\alpha;[t,T_X]}\leq B_0$$

(2) The error function $\xi:[t,T_X]\to\mathbb{R}^d$

$$\xi\left(s\right)=X_s-x-\int_t^s b\left(r,X_r\right)dr-\int_t^s\sigma\left(r,X_r\right)dg\left(r\right),\quad s\in\left[t,T_X\right],$$

satisfies

$$\begin{aligned}&(a)\qquad \left|\xi\left(s\right)\right|\leq\varepsilon\left(s-t\right),\ \text{for all } s\in\left[t,T_X\right],\\ &(b)\qquad \left|\xi\left(\tau\right)-\xi\left(s\right)\right|\leq D_0\left|\tau-s\right|^{1-\alpha},\quad \text{for all } s,\tau\in\left[t,T_X\right],\end{aligned}$$

where the constant D_0 depends only on R_0,L_{R_0}, $M_{0,T}$, M_0, L_0 T, α, β and $\Lambda_\alpha\left(g\right)$.

Remark that B_0 and D_0 are independent of ε.

The set $\mathcal{A}_\varepsilon\left(t,x\right)$ is not empty because we can find $(t,x)\in\mathcal{A}_\varepsilon\left(t,x\right)$.
$\mathcal{A}_\varepsilon\left(t,x\right)$ is an inductive set for the order relation

$$\left(T_{X_1},X_1\left(\cdot\right)\right)\preceq\left(T_{X_2},X_2\left(\cdot\right)\right)$$

defined by

$$T_{X_1} \leq T_{X_2} \text{ and } X_2(\cdot)\Big|_{[t,T_{X_1}]} = X_1(\cdot).$$

Zorn's Lemma implies that there exists a maximal element $(T^\varepsilon, X^\varepsilon) \in \mathcal{A}_\varepsilon(t,x)$. We shall prove by *reductio ad absurdum* that $T^\varepsilon = T$.

Assume that $T^\varepsilon < T$. Denote $X^\varepsilon_{T^\varepsilon} = x^\varepsilon$. We have $\|X^\varepsilon\|_{1-\alpha;[t,T_X]} \leq B_0$ and in particular

$$|x^\varepsilon| \leq B_0.$$

We know from the hypotheses that $(b(T^\varepsilon, x^\varepsilon), \sigma(T^\varepsilon, x^\varepsilon))$ is $(1-\alpha)-$fractional $g-$contingent to K in $(T^\varepsilon, x^\varepsilon)$, i.e. there exist $\bar{h}_\varepsilon > 0$ sufficiently small (for moment $0 < \bar{h}_\varepsilon < (T - T^\varepsilon) \wedge 1$), $Q^\varepsilon : [T^\varepsilon, T^\varepsilon + \bar{h}_\varepsilon] \to \mathbb{R}^d$, two constants $G_0 = G_{B_0}$, $\tilde{G}_0 = \tilde{G}_{B_0} > 0$ independent of $(T^\varepsilon, \bar{h}_\varepsilon)$ and a constant $\gamma = \gamma_{B_0} \in (0,1)$ also independent of $(T^\varepsilon, \bar{h}_\varepsilon)$ (the constants G_0,$\tilde{G}_0$, γ depend only on $R_0, L_{R_0}, M_{0,T}$, M_0, L_0 T, α, β) such that for all $s, \tau \in [T^\varepsilon, T^\varepsilon + \bar{h}_\varepsilon]$

$$|Q^\varepsilon(\tau) - Q^\varepsilon(s)| \leq G_0 |\tau - s|^{1-\alpha} \quad \text{and} \quad |Q^\varepsilon(s)| \leq \tilde{G}_0 |s - T^\varepsilon|^{1+\gamma}$$

and satisfying for all $s \in [T^\varepsilon, T^\varepsilon + \bar{h}_\varepsilon]$

$$x^\varepsilon + (s - T^\varepsilon) b(T^\varepsilon, x^\varepsilon) + \sigma(T^\varepsilon, x^\varepsilon)[g(s) - g(T^\varepsilon)] + Q^\varepsilon(s) \in K(s),$$

We set $S^\varepsilon = T^\varepsilon + \bar{h}_\varepsilon$ and we define $\hat{X}^\varepsilon : [t, S^\varepsilon] \to K$ as a extension of X^ε by

$$\hat{X}^\varepsilon(s) = \begin{cases} X^\varepsilon(s), \quad \text{if } s \in [t, T^\varepsilon], \\ x^\varepsilon + (s - T^\varepsilon) b(T^\varepsilon, x^\varepsilon) + \sigma(T^\varepsilon, x^\varepsilon)(g(s) - g(T^\varepsilon)) + Q^\varepsilon(s), \\ \hfill \text{if } s \in [T^\varepsilon, S^\varepsilon]. \end{cases}$$

We will prove that the extension $\left(S^\varepsilon, \hat{X}^\varepsilon\right) \in \mathcal{A}_\varepsilon(t,x)$.

Step 1: Clearly $\hat{X}^\varepsilon_t = x$ and $\hat{X}^\varepsilon_s \in K(s)$ for all $s \in [t,T]$.

Let us show that $\left\|\hat{X}^\varepsilon\right\|_{1-\alpha;[t,S^\varepsilon]} \leq B_0^{(1)}$ where $B_0^{(1)} \geq R_0$ and $B_0^{(1)}$ depends only on R_0, L_{R_0}, $M_{0,T}$, M_0, L_0 T, α, β and $\Lambda_\alpha(g)$.

Let $T^\varepsilon \leq s \leq \tau \leq S^\varepsilon$. Then

$$\begin{aligned} \left|\hat{X}^\varepsilon_\tau - \hat{X}^\varepsilon_s\right| &\leq |\tau - s| \, |b(T^\varepsilon, x^\varepsilon)| + \Lambda_\alpha(g)\Gamma(\alpha)|\sigma(T^\varepsilon, x^\varepsilon)| \, |\tau - s|^{1-\alpha} + |Q^\varepsilon(\tau) - Q^\varepsilon(s)| \\ &\leq |\tau - s| L_0(1 + |x^\varepsilon|) + \frac{\Lambda_\alpha(g)}{\alpha} M_{0,T}(1 + |x^\varepsilon|)|\tau - s|^{1-\alpha} + G_R|\tau - s|^{1-\alpha} \\ &\leq C_{R_0}|\tau - s|^{1-\alpha}, \end{aligned} \tag{22}$$

and $\left|\hat{X}^{\varepsilon}_{\tau}\right| \leq \left|\hat{X}^{\varepsilon}_{\tau} - \hat{X}^{\varepsilon}_{T^{\varepsilon}}\right| + |x^{\varepsilon}| \leq C_{R_0} T^{1-\alpha} + R_0$. Hence

$$\left\|\hat{X}^{\varepsilon}\right\|_{1-\alpha;[t,S^{\varepsilon}]} \leq \left\|\hat{X}^{\varepsilon}\right\|_{1-\alpha;[t,T^{\varepsilon}]} + \left\|\hat{X}^{\varepsilon}\right\|_{1-\alpha;[T^{\varepsilon},S^{\varepsilon}]}$$
$$\leq B_0 + C_{R_0} T^{1-\alpha} + R_0 + C_{R_0} \overset{def}{=} B_0^{(1)} \; .$$

Step 2: the error function.

Let the error functions $\xi^{\varepsilon} : [t, T^{\varepsilon}] \to \mathbb{R}^d$ and $\hat{\xi}^{\varepsilon} : [t, S^{\varepsilon}] \to \mathbb{R}^d$,

$$\xi^{\varepsilon}(s) = X^{\varepsilon}_s - x - \int_t^s b(r, X^{\varepsilon}_r)\, dr - \int_t^s \sigma(r, X^{\varepsilon}_r)\, dg(r) \, .$$
$$\hat{\xi}^{\varepsilon}(s) = \hat{X}^{\varepsilon}_s - x - \int_t^s b\left(r, \hat{X}^{\varepsilon}_r\right) dr - \int_t^s \sigma\left(r, \hat{X}^{\varepsilon}_r\right) dg(r) \, .$$

Clearly $\left|\hat{\xi}^{\varepsilon}(s)\right| = |\xi^{\varepsilon}(s)| \leq \varepsilon (s-t)$ for all $s \in [t, T^{\varepsilon}]$.

Let $s \in [T^{\varepsilon}, S^{\varepsilon}]$. Using Lemma 8 (the inequalities (16) with $Y_r = \hat{X}^{\varepsilon}_r$ and $t = T^{\varepsilon}$, we have

$$\left|\hat{\xi}^{\varepsilon}(s)\right| \leq \left| x^{\varepsilon} - x - \int_t^{T^{\varepsilon}} b(r, X^{\varepsilon}_r)\, dr - \int_t^{T^{\varepsilon}} \sigma(r, X^{\varepsilon}_r)\, dg(r) \right|$$
$$+ \left| (s - T^{\varepsilon})\, b(T^{\varepsilon}, x^{\varepsilon}) + \sigma(T^{\varepsilon}, x^{\varepsilon})(g(s) - g(T^{\varepsilon})) - \int_{T^{\varepsilon}}^s b\left(r, \hat{X}^{\varepsilon}_r\right) dr - \int_{T^{\varepsilon}}^s \sigma\left(r, \hat{X}^{\varepsilon}_r\right) dg(r) \right|$$
$$+ |Q^{\varepsilon}(s)|$$
$$\leq \varepsilon (T^{\varepsilon} - t) + \left| \int_{T^{\varepsilon}}^s \left[b(T^{\varepsilon}, x^{\varepsilon}) - b\left(r, \hat{X}^{\varepsilon}_r\right)\right] dr + \int_{T^{\varepsilon}}^s \left[\sigma(T^{\varepsilon}, x^{\varepsilon}) - \sigma\left(r, \hat{X}^{\varepsilon}_r\right)\right] dg(r) \right|$$
$$+ \tilde{G}_0 (s - T^{\varepsilon})^{1+\gamma}$$
$$\leq \varepsilon (T^{\varepsilon} - t) + C^{(1)}_{R_0} (s - T^{\varepsilon})^{2-\alpha} + C^{(2)}_{R_0} (s - T^{\varepsilon})^{1+\min\{\beta-\alpha, 1-2\alpha\}} + \tilde{G}_0 (s - T^{\varepsilon})^{1+\gamma}$$
$$\leq \varepsilon (T^{\varepsilon} - t) + \varepsilon (s - T^{\varepsilon}) = \varepsilon (s - t)$$

for $\bar{h}_{\varepsilon}$ sufficiently small such that

$$C^{(1)}_{R_0} \bar{h}_{\varepsilon}^{1-\alpha} + C^{(2)}_{R_0} \bar{h}_{\varepsilon}^{\min\{\beta-\alpha, 1-2\alpha\}} + \tilde{G}_0 \bar{h}_{\varepsilon}^{\gamma} \leq \varepsilon.$$

Hence

$$\left|\hat{\xi}^{\varepsilon}(s)\right| \leq \varepsilon (s - t) \;\text{ for all } s \in [t, S^{\varepsilon}] \, .$$

Let now $T^{\varepsilon} \leq \tau < s \leq S^{\varepsilon}$. Then by Lemma 8 (the inequalities (17) with $Y_r = \hat{X}^{\varepsilon}_r$, $t = T^{\varepsilon}$ and $x^{\varepsilon} = \hat{X}^{\varepsilon}_{T^{\varepsilon}}$ we have

$$\left|\hat{\xi}^{\varepsilon}(s) - \hat{\xi}^{\varepsilon}(\tau)\right|$$
$$\leq \left| \int_{\tau}^s \left[b(T^{\varepsilon}, x^{\varepsilon}) - b\left(r, \hat{X}^{\varepsilon}_r\right)\right] dr + \int_{\tau}^s \left[\sigma(T^{\varepsilon}, x^{\varepsilon}) - \sigma\left(r, \hat{X}^{\varepsilon}_r\right)\right] dg_r \right| + |Q^{\varepsilon}(s) - Q^{\varepsilon}(\tau)|$$

$$\leq C_{R_0}^{(3)}(s-\tau)+C_{R_0}^{(4)}(s-\tau)^{1-\alpha}+G_0|\tau-s|^{1-\alpha}$$
$$\leq C_{R_0}(s-\tau)^{1-\alpha}.$$

From the definition of $\mathcal{A}_\varepsilon(t,x)$, for $\tau,s\in[t,T^\varepsilon]$

$$\left|\hat{\xi}^\varepsilon(s)-\hat{\xi}^\varepsilon(\tau)\right|=|\xi^\varepsilon(s)-\xi^\varepsilon(\tau)|\leq D_0\ (s-\tau)^{1-\alpha},\ \ \forall\ \tau,s\in[t,T^\varepsilon].$$

We conclude

$$\left|\hat{\xi}^\varepsilon(s)-\hat{\xi}^\varepsilon(\tau)\right|\leq(C_{R_0}\vee D_0)|\tau-s|^{1-\alpha},\ \ \forall\ \tau,s\in[t,S^\varepsilon].$$

We arrived to prove that $\left(S^\varepsilon,\hat{X}^\varepsilon\right)$ is proper extension of $(T^\varepsilon,X^\varepsilon)$, that contradicts the maximallity of $(T^\varepsilon,X^\varepsilon)$ in $\mathcal{A}_\varepsilon(t,x)$. Therefore $T^\varepsilon=T$.

Let $(T,\ X^\varepsilon)$ be a maximal element of $\mathcal{A}_\varepsilon(t,x)$. Then from the definition of $\mathcal{A}_\varepsilon(t,x)$ we have $X_t^\varepsilon=x,\ X_s^\varepsilon\in K(s)$ for all $s\in[t,T]$, and there exists a positive constant $B_0\geq R_0$ depending only on $R_0,L_{R_0},\ M_{0,T},\ M_0,\ L_0\ T,\ \alpha,\ \beta$ and $\Lambda_\alpha(g)$, such that

$$\|X^\varepsilon\|_{1-\alpha;[t,T]}\leq B_0.$$

The error function $\xi^\varepsilon:[t,T]\to\mathbb{R}^d$,

$$\xi^\varepsilon(s)=X_s^\varepsilon-x-\int_t^s b(r,X_r^\varepsilon)\,dr-\int_t^s\sigma(r,X_r^\varepsilon)\,dg(r)\tag{23}$$

satisfies

$$\begin{array}{ll}(a) & |\xi^\varepsilon(s)|\leq\varepsilon(s-t),\ \text{ for all } s\in[t,T],\\ (b) & |\xi^\varepsilon(\tau)-\xi^\varepsilon(s)|\leq D_0|\tau-s|^{1-\alpha},\quad \text{for all } s,\tau\in[t,T],\end{array}$$

where the constant D_0 depends only on $R_0,L_{R_0},\ M_{0,T},\ M_0,\ L_0\ T,\ \alpha,\ \beta$ and $\Lambda_\alpha(g)$.

We now estimate $\|\xi^\varepsilon\|_{\alpha,\infty;[t,T]}$. Let $\lambda\geq0$. Then

$$\|\xi^\varepsilon\|_{\alpha,\lambda;[t,T]}\leq\|\xi^\varepsilon\|_{\alpha,\infty;[t,T]}$$
$$=\sup_{s\in[t,T]}\left\{|\xi^\varepsilon(s)|+\int_t^s\frac{|\xi^\varepsilon(s)-\xi^\varepsilon(r)|}{(s-r)^{\alpha+1}}dr\right\}$$
$$\leq\varepsilon(T-t)+\sup_{s\in[t,T]}\int_t^s|\xi_s^\varepsilon-\xi_r^\varepsilon|^{\frac12-\alpha}\frac{|\xi_s^\varepsilon-\xi_r^\varepsilon|^{\frac12+\alpha}}{(s-r)^{\alpha+1}}dr$$
$$\leq\varepsilon(T-t)+[2\varepsilon(T-t)]^{\frac12-\alpha}\sup_{s\in[t,T]}\int_t^s\frac{D_0^{\frac12+\alpha}(s-r)^{(1-\alpha)\left(\frac12+\alpha\right)}}{(s-r)^{1+\alpha}}dr$$
$$=\varepsilon(T-t)+[2\varepsilon(T-t)]^{\frac12-\alpha}\frac{D_0^{\frac12+\alpha}}{\left(\frac12-\alpha\right)(1+\alpha)}(T-t)^{\left(\frac12-\alpha\right)(1+\alpha)}$$

$$\leq C_{R_0}\ \varepsilon^{\frac{1}{2}-\alpha}\ .$$

It remains now to prove that the limit of the sequence X^{ε} exists as $\varepsilon \to 0$ and this limit is a solution to the differential equation (19)
Let $0<\varepsilon,\eta \leq 1$. Using the estimates (12) and (15), we get

$$\begin{aligned}
&\|X^{\varepsilon}-X^{\eta}\|_{\alpha,\lambda;[t,T]}\\
&\leq \left\|F_{t,\cdot}^{(b)}\left(X^{\varepsilon}\right)-F_{t,\cdot}^{(b)}\left(X^{\eta}\right)\right\|_{\alpha,\lambda;[t,T]}+\left\|G_{t,\cdot}^{(\sigma)}\left(X^{\varepsilon}\right)-G_{t,\cdot}^{(\sigma)}\left(X^{\eta}\right)\right\|_{\alpha,\lambda;[t,T]}+\|\xi^{\varepsilon}-\xi^{\eta}\|_{\alpha,\lambda;[t,T]}\\
&\leq \frac{C_R^{(b3)}}{\lambda^{\alpha}}\|X^{\varepsilon}-X^{\eta}\|_{\alpha,\lambda;[t,T]}+\frac{C_R^{(\sigma3)}\Lambda_{\alpha}(g)}{\lambda^{1-2\alpha}}\left(1+\Delta_{[t,T]}\left(X^{\varepsilon}\right)+\Delta_{[t,T]}\left(X^{\eta}\right)\right)\|X^{\varepsilon}-X^{\eta}\|_{\alpha,\lambda;[t,T]}\\
&\qquad +C_{R_0}\ \varepsilon^{\frac{1}{2}-\alpha}+C_{R_0}\ \eta^{\frac{1}{2}-\alpha}
\end{aligned}$$

Since $\|X^{\varepsilon}\|_{1-\alpha;[t,T]}\leq B_0$, $0<\alpha<\frac{\delta}{1+\delta}$, then

$$\begin{aligned}
\Delta_{[t,T]}\left(X^{\varepsilon}\right)=\sup\nolimits_{r\in[t,T]}\int_t^r\frac{|X_r^{\varepsilon}-X_s^{\varepsilon}|^{\delta}}{(r-s)^{1+\alpha}}\quad &\leq B_0\sup\nolimits_{r\in[t,T]}\int_t^r\frac{(r-s)^{(1-\alpha)\delta}}{(r-s)^{\alpha+1}}ds\\
&\leq B_0\frac{T^{\delta-\alpha(1+\delta)}}{\delta-\alpha\left(1+\delta\right)}\\
&\leq \frac{B_0\left(1+T\right)}{\delta-\alpha\left(1+\delta\right)}
\end{aligned}$$

and therefore

$$\begin{aligned}
\|X^{\varepsilon}-X^{\eta}\|_{\alpha,\lambda;[t,T]}\leq \frac{C_{R_0}^{(3)}}{\delta-\alpha\left(1+\delta\right)}\left(\frac{1}{\lambda^{\alpha}}+\frac{1}{\lambda^{1-2\alpha}}\right)\|X^{\varepsilon}-X^{\eta}\|_{\alpha,\lambda;[t,T]}\\
+C_{R_0}\ \varepsilon^{\frac{1}{2}-\alpha}+C_{R_0}\ \eta^{\frac{1}{2}-\alpha}
\end{aligned}$$

Let $\lambda=\bar{\lambda}\geq 1$ such that

$$\frac{C_{R_0}^{(3)}}{\delta-\alpha\left(1+\delta\right)}\left(\frac{1}{\bar{\lambda}^{\alpha}}+\frac{1}{\bar{\lambda}^{1-2\alpha}}\right)\leq\frac{1}{2}.$$

We deduce

$$\|X^{\varepsilon}-X^{\eta}\|_{\infty;[t,T]}\leq e^{\bar{\lambda}T}\|X^{\varepsilon}-X^{\eta}\|_{\alpha,\bar{\lambda};[t,T]}\leq 2C_{R_0}e^{\bar{\lambda}T}\ \left(\varepsilon^{\frac{1}{2}-\alpha}+\ \eta^{\frac{1}{2}-\alpha}\right).$$

Hence there exists $X^{t,x}$ such that $X^{\varepsilon}\to X^{t,x}$ in $C\left([t,T];K\right)$ and $X^{\varepsilon}\to X^{t,x}$ in $W^{\alpha,\infty}(t,T;\mathbb{R}^d)$ as $\varepsilon\to 0$. Since for all $s,\tau\in[t,T]$:

$$\|X^{\varepsilon}\|_{\infty;[t,T]}+\frac{|X_s^{\varepsilon}-X_{\tau}^{\varepsilon}|}{|s-\tau|^{1-\alpha}}\leq B_0$$

then passing to limit as $\varepsilon \to 0$ we obtain

$$\left\|X^{t,x}\right\|_{1-\alpha;[t,T]} \leq B_0.$$

Since $X_s^\varepsilon \in K(s)$ for all $s \in [t,T]$, clearly follows, as $\varepsilon \to 0$, that

$$X_s \in K(s) \quad \text{for all} \quad s \in [t,T].$$

Passing to limit in (23) we infer that $X^{t,x}$ is a solution of the differential equation (19) starting at t from x and evolving in the tube $\{(s,y) : s \in [t,T],\ y \in K(s)\}$.
The proof is complete.

■

5 Proof of the main result

Let be fixed a parameter $1/2 < H < 1$. Consider $B = \{B_t, t \in [0,T]\}$ be a $\mathbb{R}^k$ valued fractional Brownian motion with parameter H defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. From (1)it follows that

$$\mathbb{E}\left(|B_t - B_s|^2\right) = |t-s|^{2H}$$

and as a consequence, for any $p \geq 1$,

$$\|B_t - B_s\|_p = (\mathbb{E}(|B_t - B_s|^p))^{1/p} = c_p\,|t-s|^H.$$

It is known that the random variable

$$G = \frac{1}{\Gamma(1-\alpha)} \sup_{t<s<r<T} \left|\left(D_{r-}^{1-\alpha} B_{r-}\right)(s)\right|$$

has moments of all order. As a consequence, As a consequence, if $u = \{u_t,\ t \in [0,T]\}$ is a stochastic process whose trajectories belong to the space $W^{\alpha,1}(0,T;\mathbb{R}^{d\times k})$, with $1 - H < \alpha < \frac{1}{2}$, the pathwise integral $\int_0^T u_s dB_s$ exists in the sense of Definition 2 and we have the estimate

$$\left|\int_0^T u_s dB_s\right| \leq G\ \|u\|_{\alpha,1}.$$

Moreover, if the trajectories of the process u belong to the space $W^{\alpha,\infty}(0,T;\mathbb{R}^{d\times k})$, then the indefinite integral $U_t = \int_0^t u_s dB_s$ is Holder continuous of order $1-\alpha$, and the estimates from Proposition 3 hold.
Proof of the Theorem 12. Considering the previous observations, the solution follows directly from the deterministic Theorem 16.

■

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